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World-line quantization of a reciprocally invariant system

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Abstract

We present the world-line quantization of a system invariant under the symmetries of reciprocal relativity (pseudo-unitary transformations on 'phase-space coordinates' $(x^\mu(\tau), p^\mu(\tau))$ which preserve the Minkowski metric and the symplectic form, and global shifts in these coordinates, together with coordinate-dependent transformations of an additional compact phase coordinate, $\theta(\tau)$). The action is that of free motion over the corresponding Weyl–Heisenberg group. Imposition of the first class constraint, the generator of local time reparametrizations, on physical states enforces identification of the world-line cosmological constant with a fixed value of the quadratic Casimir of the quaplectic symmetry group $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$, the semi-direct product of the pseudo-unitary group with the Weyl–Heisenberg group (the central extension of the global translation group, with central extension associated with the phase variable $\theta(\tau)$). The spacetime spectrum of physical states is identified. Even though for an appropriate range of values the restriction enforced by the cosmological constant projects out negative norm states from the physical gauge invariant spectrum, leaving over spin zero states

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only, in this purely bosonic setting the mass-squared spectrum is continuous over the entire real line and thus includes a tachyonic branch as well.

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1. Introduction

Born reciprocity [1] is based on the observation of the apparent exchangeability of 'position' and 'momentum' in much of the formalism of classical and quantum physics, and seeks to elevate this equivalence to a fundamental principle. The idea of Born [1, 2] and Green [3, 4] was to formalize this by extending the Minkowski metric of Einstein's special relativity to an invariant metric on 'phase-space coordinates'

$$\mathrm{d}\ell^2 = \mathrm{d}s^2 + \frac{c^4}{b^2}\,\mathrm{d}m^2 = \mathrm{d}x^\mu\,\mathrm{d}x_\mu + \frac{c^2}{b^2}\,\mathrm{d}p^\mu\,\mathrm{d}p_\mu, \tag{1}$$

where

$$\begin{aligned} &x^\mu = (ct, \vec{x}), && p^\mu = \left(\frac{E}{c}, \vec{p}\right), \\ &\eta_{\mu\nu} = \mathrm{diag}(- + + \cdots +), && \mu = 0, 1, 2, \ldots, D-1, \end{aligned} \tag{2}$$

which can be seen as introducing a new fundamental constant, here a maximal universal unit of force $b > 0$ (which can also be thought of in terms of fundamental constants of acceleration, or length, or time, depending on the interpretation). Born and Green sought reciprocally invariant 'master' equations whose zeros were interpreted via multi-mass relativistic wave equations for the meson spectrum. The idea of reciprocity has found resonance with various attempts to generalize the framework for the fundamental interactions—for example, in the context of string and M-theory [5, 6], in the guise of bi-crossproduct algebras and physics at the Planck scale [7], in 'two-time' formulations [8], or in ad hoc 'noncommutative geometry' extensions of perturbative field theory [9].

Born–Green reciprocity can be viewed [10–13] as an alternative paradigm for generalized wave equations, which specify unitary irreducible representations of the full symmetry group, in the same way that relativistic wave equations establish unitary irreducible representations of the Poincaré group in four dimensions. It can be argued [10–13] that the appropriate invariance group is the so-called quaplectic group $Q(3,1) \cong U(3,1) \ltimes H(4)$, or more generally in D spacetime dimensions, the group $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$ of reciprocal relativity, the semi-direct product of the pseudo-unitary group of linear transformations between x^μ and p^μ which preserve both the extended metric $\mathrm{d}\ell^2$ and the symplectic form, with the Weyl–Heisenberg group. The Wigner–Mackey method of induced representations can be applied for this case, and $b \to \infty$ contraction limits of the appropriate generalized reciprocally invariant wave equations should collapse to the standard relativistic wave equations of particle physics, as, for example, solutions of the massive Klein–Gordon equation can be seen as going over to the Galilean invariant nonrelativistic wavefunctions in the $c \to \infty$ limit [14].

In this paper, we study the alternative route to particle equations of motion via Hamiltonian quantization of constrained systems on the world-line [15]. We present the world-line quantization of a system invariant under the symmetries of Born–Green reciprocity, realized as transformations on 'phase-space coordinates' $(x^\mu(\tau), p^\mu(\tau))$ on the world-line, for an action, considered in section 2, which is that of free motion on the associated Weyl–Heisenberg group, a guarantee from the outset for full quaplectic invariance. These transformations are global Lorentz variations, $x^\mu \rightarrow x^\mu + \omega^\mu{}_\nu x^\nu$, $p^\mu \rightarrow p^\mu + \omega^\mu{}_\nu p^\nu$ (in infinitesimal form, with $\omega_{\mu\nu} = -\omega_{\nu\mu}$), global quaplectic 'boosts', $x^\mu \rightarrow x^\mu + \alpha^\mu{}_\nu p^\nu/b$, $p^\mu \rightarrow p^\mu - b\alpha^\mu{}_\nu x^\nu$ (in infinitesimal form, with $\alpha_{\mu\nu} = \alpha_{\nu\mu}$), comprising the pseudo-unitary group $U(D-1,1)$, together with global translations in both variables x^μ and p^μ. The action contains terms quadratic in velocities, and also linearly coupled terms, in such a way that both the extended metric $\mathrm{d}\ell^2/\mathrm{d}\tau^2$, and the symplectic form $p_\mu(\tau)\,\mathrm{d}x^\mu(\tau)/\mathrm{d}\tau - x_\mu(\tau)\,\mathrm{d}p^\mu(\tau)/\mathrm{d}\tau$, are evident. The latter occurs in a term which plays the role of a minimal Landau-type coupling in the kinetic energy of a further scalar variable $\theta(\tau)$, interpreted as a phase (S^1) degree of freedom associated with the unit operator of the Weyl–Heisenberg algebra. In this way global translations in x^μ and p^μ, which do not leave the symplectic form invariant, are compensated by appropriate x- and p-dependent transformations of the phase θ, so that translation invariance is restored overall. The full algebra of Noether charges thus comprises all of the conserved charges which generate these transformations. These include the generator, $\mathcal{P}^\mu$, of spacetime translations in x^μ to be interpreted as the conserved energy–momentum of the system, as well as the generator, $L_T^{\mu\nu}$, of spacetime Lorentz transformations in x^μ and p^μ, to be interpreted as the total relativistic angular-momentum of the system. Both these quantities generate the Poincaré invariance of the system, of which the representation realized by the space of quantum states determines the mass and spin content of the quantized dynamics. Whether at the classical or quantum level the conserved charges for translations in x^μ and p^μ, $\mathcal{P}^\mu$ and $\mathcal{X}^\mu$, do not commute, but their algebra possesses a central extension given by the conserved generator Π_θ of translations in the phase $\theta(\tau)$. In any of the superselection sectors with a fixed nonvanishing discrete eigenvalue of this generator (given the compactness of the θ part of configuration space), $\mathcal{P}^\mu$ and $\mathcal{X}^\mu$ thus fulfil a Heisenberg algebra, and can be identified with the physical energy–momentum and possibly even the spacetime position, respectively. The remaining Noether charges are the sum of quadratic combinations of these charges, plus quadratic combinations of non-conserved auxiliary Heisenberg algebra generators, $\mathbb{X}^\mu$ and $\mathbb{P}^\mu$, which commute with $\mathcal{P}^\mu$ and $\mathcal{X}^\mu$. The full symmetry algebra is thus the semi-direct product of the conserved Heisenberg algebra, and the homogeneous charges, which indeed together generate the Lie algebra of the quaplectic group, $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$. These considerations are detailed in section 3.

In section 4, the model is extended to include local time reparametrizations. The associated Noether symmetry, the generator of local world-line gauge transformations, is the extended Hamiltonian and becomes a first class constraint. Following the Dirac quantization procedure, imposition of the first class constraint on physical states enforces the condition that the cosmological constant term, allowed by the reparametrization invariant coupling of the system to the world-line metric, must be identified with a discrete fixed eigenvalue of the quadratic Casimir invariant of the quaplectic algebra, in line with well-understood features of Hamiltonian quantization [16]. The decomposition of such irreducible representations of the full quaplectic group with respect to the physical Poincaré group $IO(D-1,1) \cong O(D-1,1) \ltimes T(D)$ is discussed, where the generators $\mathcal{P}^\mu$ are identified with the standard energy–momentum operators, the generators of spacetime translations. From this analysis follows the spectrum of physical states classified according to the eigenspectrum of the Lorentz covariant mass-squared and generalized Pauli–Lubanski operators. A noteworthy result is that an appropriate choice of cosmological constant projects out any negative norm state from the physical spectrum. Nevertheless in the present formulation of such reciprocally invariant dynamics involving bosonic degrees of freedom only the mass spectrum is continuous and always contains spacelike, namely tachyonic states. Conclusions and possible extensions of this work are addressed in section 5, while an appendix briefly outlines the rationale behind the choice of action used in the next section.

2. Dynamics and symmetries

2.1. The action

For reasons discussed in the appendix, the degrees of freedom of the system to be considered are the spacetime and conjugate coordinates, $x^\mu(\tau)$ and $p^\mu(\tau)$, as well as an angular variable, $\theta(\tau)$, taking its values in the periodic range $0 \leqslant \theta \leqslant 2\pi$. Spacetime indices $\mu, \nu = 0, 1, 2, \ldots, D-1$ are raised and lowered using the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(- + + \cdots +)$. The inner product defined by that metric is denoted with a dot, namely, $x \cdot p = x^\mu p_\mu$.

The dynamics of the configuration space variables $(x^\mu(\tau), p^\mu(\tau), \theta(\tau))$ is taken to follow from the action principle:

$$S[x^\mu, p^\mu, \theta] = \int \mathrm{d}\tau L, \qquad L = \frac{1}{2N_0}[\dot{x}^2 + \kappa_0 \dot{p}^2] + \frac{1}{2N_0}\frac{\alpha\kappa_0}{\lambda_0}[\dot{\theta} - \lambda_0(\dot{x}\cdot p - x \cdot \dot{p})]^2. \quad (3)$$

In this expression, $N_0 > 0$, $\kappa_0 = c^2/b^2 \geqslant 0$, and α and λ_0, with $\alpha\lambda_0 > 0$, are normalization factors specifying the physical properties of the system. Their dimension is such that the action S has the dimension of $\hbar$, namely $M \cdot L^2 \cdot T^{-1}$. Since we shall not work in 'natural' units but take the time evolution variable τ to be dimensionless, the physical dimensions of these parameters are as follows:

$$[N_0] = M^{-1} \cdot T, \qquad [\kappa_0] = M^{-2} \cdot T^2, \qquad [\lambda_0] = M^{-1} \cdot L^{-2} \cdot T, \qquad [\alpha] = M \cdot T^{-1}. \quad (4)$$

As explained in the appendix, this choice of action follows from considering free motion on the Weyl–Heisenberg group associated with the variables (x^μ, p^μ). The rationale behind such a choice is that, on the one hand, it readily guarantees from the outset a dynamics which is invariant under the full quaplectic group, and on the other hand, it generalizes within this context the situation for the ordinary relativistic particle which corresponds to motion on the group of spacetime translations coupled to a world-line metric in a reparametrization invariant manner. Thus a relativistic particle dynamics invariant under the reciprocal symmetry of the quaplectic group may likewise be constructed by coupling the above action to a world-line metric in a reparametrization invariant manner, as done in section 4. In the present section, the symmetries of the above action are described and its dynamics understood in the next section. The system coupled to a world-line metric may then readily be solved based on the general considerations of [16].

By construction, the action (3) possesses a series of global symmetries generated by the corresponding Noether charges. First one has invariance under translations in the spacetime coordinates x^μ,

$$x^\mu(\tau) \to x^\mu(\tau) + a^\mu, \qquad p^\mu(\tau) \to p^\mu(\tau), \qquad \theta(\tau) \to \theta(\tau) - \lambda_0 a^\mu p_\mu(\tau), \quad (5)$$

of which the Noether charges are denoted by $\mathcal{P}^\mu$, since they measure the conserved total energy–momentum of the system and indeed possess the appropriate physical dimension. Next one has a dual symmetry, namely invariance under translations in the dual configuration space coordinates p^μ,

$$x^\mu(\tau) \to x^\mu(\tau), \qquad p^\mu(\tau) \to p^\mu(\tau) + k^\mu, \qquad \theta(\tau) \to \theta(\tau) + \lambda_0 k^\mu x_\mu(\tau), \quad (6)$$

of which the Noether charges are denoted by $\mathcal{X}^\mu$, having indeed the same physical dimensions as the spacetime coordinates x^μ. The charges $\mathcal{X}^\mu$ and $\mathcal{P}^\mu$ are the generators of the Heisenberg subgroup $H(D)$ of the full quaplectic symmetry $Q(D-1, 1) \cong U(D-1, 1) \ltimes H(D)$.

In infinitesimal form, the Lorentz symmetry

$$\begin{aligned} &x^{\mu}(\tau) \to \Lambda^{\mu}{}_{\nu} x^{\nu}(\tau), && p^{\mu}(\tau) \to \Lambda^{\mu}{}_{\nu} p^{\nu}(\tau), \\ &\theta(\tau) \to \theta(\tau), && \eta_{\rho\sigma} \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}, \end{aligned} \tag{7}$$

reduces to the transformations

$$\begin{aligned} &x^{\mu}(\tau) \to x^{\mu}(\tau) + \omega^{\mu}{}_{\nu} x^{\nu}(\tau), && p^{\mu}(\tau) \to p^{\mu}(\tau) + \omega^{\mu}{}_{\nu} p^{\nu}(\tau), \\ &\theta(\tau) \to \theta(\tau), && \omega_{\nu\mu} = -\omega_{\mu\nu}, \end{aligned} \tag{8}$$

with the total relativistic angular-momentum $L_T^{\mu\nu} = -L_T^{\nu\mu}$ as the conserved Noether charge. Likewise the dual symmetry corresponds to the symplectic transformations which in infinitesimal form read

$$\begin{aligned} &x^{\mu}(\tau) \to x^{\mu}(\tau) + \sqrt{\kappa_0}\alpha^{\mu}{}_{\nu} p^{\nu}(\tau), && p^{\mu}(\tau) \to p^{\mu}(\tau) - \frac{1}{\sqrt{\kappa_0}}\alpha^{\mu}{}_{\nu} x^{\nu}(\tau), \\ &\theta(\tau) \to \theta(\tau), && \alpha_{\nu\mu} = \alpha_{\mu\nu}, \end{aligned} \tag{9}$$

and possess conserved Noether charges denoted by $M^{\mu\nu}$ with $M^{\nu\mu} = M^{\mu\nu}$. The charges $L_T^{\mu\nu}$ and $M^{\mu\nu}$ are the generators of the pseudo-unitary subgroup $U(D-1,1)$ of the full quaplectic symmetry $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$.

Finally, invariance under translations in the angular variable $\theta(\tau)$,

$$x^{\mu}(\tau) \to x^{\mu}(\tau), \qquad p^{\mu}(\tau) \to p^{\mu}(\tau), \qquad \theta(\tau) \to \theta(\tau) + \theta_0, \tag{10}$$

possess a Noether charge denoted by Q_θ.

Rather than giving here the expressions, whether within the Lagrangian or Hamiltonian formalisms, for all these quantities and their equations of motion, it turns out to be particularly useful to introduce a complex parametrization for the dynamics in which the pairs of variables (x^μ, p^μ) for each spacetime index $\mu = 0, 1, 2, \ldots, D-1$ are combined into a single complex-valued quantity. This choice of representation of the dynamics is also perfectly adapted to its inherent quaplectic symmetry properties.

2.2. *The complex parametrization*

By introducing the complex parametrization of configuration space in its spacetime sector

$$z^{\mu} = \tfrac{1}{\sqrt{2}}[x^{\mu} + \mathrm{i}\sqrt{\kappa_0} p^{\mu}], \tag{11}$$

the action (3) reads (a bar on top of a quantity denotes of course its complex conjugate)

$$S[z^{\mu}, \bar{z}^{\mu}, \theta] = \int \mathrm{d}\tau L, \qquad L = \frac{1}{N_0}\dot{\bar{z}} \cdot \dot{z} + \frac{1}{2N_0}\frac{\alpha\kappa_0}{\lambda_0}\left[\dot{\theta} + \mathrm{i}\frac{\lambda_0}{\sqrt{\kappa_0}}(\dot{\bar{z}} \cdot z - \bar{z} \cdot \dot{z})\right]^2. \tag{12}$$

The previously discussed global symmetries in infinitesimal form are then expressed as

$$z^{\mu}(\tau) \to z^{\mu}(\tau) + \Omega^{\mu}{}_{\nu} z^{\nu}(\tau) + A^{\mu}, \qquad \theta(\tau) \to \theta(\tau) + \theta_0 - \mathrm{i}\frac{\lambda_0}{\sqrt{\kappa_0}}[\bar{z}(\tau) \cdot A - z(\tau) \cdot \bar{A}], \tag{13}$$

with the correspondences

$$\Omega_{\mu\nu} = \omega_{\mu\nu} - \mathrm{i}\alpha_{\mu\nu}, \qquad \overline{\Omega}_{\mu\nu} = -\Omega_{\nu\mu}, \qquad A^{\mu} = \tfrac{1}{\sqrt{2}}[a^{\mu} + \mathrm{i}\sqrt{\kappa_0} k^{\mu}]. \tag{14}$$

In this form, it is clear that the global symmetry group of the system, namely the so-called quaplectic group [10–12], is indeed isomorphic to $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$. The associated Noether charges are to be denoted by, respectively for the symmetry parameters A_μ, $\Omega_{\mu\nu}$ and θ_0,

$$Q^\mu, \qquad Q^{\mu\nu} = -\overline{Q}^{\nu\mu}, \qquad Q_\theta, \tag{15}$$

with the following correspondences with the previous notations,

$$Q^\mu = \frac{1}{\sqrt{2}}\left[\mathcal{P}^\mu - \frac{\mathrm{i}}{\sqrt{\kappa_0}}\mathcal{X}^\mu\right], \qquad Q^{\mu\nu} = \frac{1}{2}L_T^{\mu\nu} + \frac{1}{2}\mathrm{i}M^{\mu\nu}. \tag{16}$$

3. Hamiltonian formulation and canonical quantization

3.1. The Noether algebra

Within the Hamiltonian formulation of the dynamics, momenta canonically conjugate to the configuration variables x^μ, p^μ, z^μ and θ are denoted, respectively, by Π_x^μ, Π_p^μ, Π^μ and Π_θ, with their canonical equal time Poisson brackets. Henceforth we shall already consider the canonically quantized system, in which these variables are operators of which the commutation relations are given by the result of the corresponding Poisson brackets multiplied by $\mathrm{i}\hbar$ (we shall refrain from introducing a notation distinguishing between operators and their classical counterparts, but the difference should be clear from the context and be kept in mind). Hence the quantized dynamics is realized as a representation of the following general tensor product of Heisenberg algebras:

$$\left[x^\mu, \Pi_x^\nu\right] = \mathrm{i}\hbar\eta^{\mu\nu}, \qquad \left[p^\mu, \Pi_p^\nu\right] = \mathrm{i}\hbar\eta^{\mu\nu}, \qquad [z^\mu, \Pi^\nu] = \mathrm{i}\hbar\eta^{\mu\nu}, \qquad [\theta, \Pi_\theta] = \mathrm{i}\hbar. \tag{17}$$

The correspondence between the quantities Π_x^μ, Π_p^μ and Π^μ is such that

$$\Pi^\mu = \frac{1}{\sqrt{2}}\left[\Pi_x^\mu - \frac{\mathrm{i}}{\sqrt{\kappa_0}}\Pi_p^\mu\right]. \tag{18}$$

Note also that as operators, a quantity with a bar on top stands for the adjoint of that operator, namely, $\overline{z}^\mu = z^{\mu\dagger}$, with respect to the implicit inner product on the space of quantum states for which the basic operators x^μ, p^μ, θ and their conjugate momenta Π_x^μ, Π_p^μ and Π_θ are Hermitian and self-adjoint.

The identification of the previously discussed Noether charges in terms of these quantities is readily achieved, leading to the expressions,

$$Q^\mu = \Pi^\mu - \mathrm{i}\frac{\lambda_0\Pi_\theta}{\sqrt{\kappa_0}}\overline{z}^\mu, \qquad Q^{\mu\nu} = z^\mu\Pi^\nu - \overline{z}^\nu\overline{\Pi}^\mu, \qquad Q_\theta = \Pi_\theta. \tag{19}$$

Separating the real and imaginary parts of Q^μ and $Q^{\mu\nu}$, one also finds

$$\mathcal{P}^\mu = \Pi_x^\mu - \lambda_0\Pi_\theta p^\mu, \qquad \mathcal{X}^\mu = \Pi_p^\mu + \lambda_0\Pi_\theta x^\mu, \tag{20}$$

$$L_T^{\mu\nu} = x^\mu\Pi_x^\nu - x^\nu\Pi_x^\mu + p^\mu\Pi_p^\nu - p^\nu\Pi_p^\mu, \tag{21}$$

$$M^{\mu\nu} = \sqrt{\kappa_0}\left[p^\mu\Pi_x^\nu + p^\nu\Pi_x^\mu\right] - \frac{1}{\sqrt{\kappa_0}}\left[x^\mu\Pi_p^\nu + x^\nu\Pi_p^\mu\right]. \tag{22}$$

In view of these expressions, let us also introduce the dual combinations

$$\mathbb{Q}^\mu = \Pi^\mu + \mathrm{i}\frac{\lambda_0\Pi_\theta}{\sqrt{\kappa_0}}\overline{z}^\mu = \frac{1}{\sqrt{2}}\left[\mathbb{P}^\mu - \frac{\mathrm{i}}{\sqrt{\kappa_0}}\mathbb{X}^\mu\right], \tag{23}$$

with thus

$$\mathbb{P}^\mu = \Pi_x^\mu + \lambda_0 \Pi_\theta p^\mu, \qquad \mathbb{X}^\mu = \Pi_p^\mu - \lambda_0 \Pi_\theta x^\mu. \tag{24}$$

Note that in terms of these variables one may write

$$Q^{\mu\nu} = \tfrac{1}{2}[z^\mu Q^\nu - \bar{z}^\nu \overline{Q}^\mu] + \tfrac{1}{2}[z^\mu \mathbb{Q}^\nu - \bar{z}^\nu \overline{\mathbb{Q}}^\mu], \tag{25}$$

as well as

$$L_T^{\mu\nu} = L_{\text{orbital}}^{\mu\nu} + [p^\mu \mathbb{X}^\nu - p^\nu \mathbb{X}^\mu], \qquad L_{\text{orbital}}^{\mu\nu} = x^\mu \mathcal{P}^\nu - x^\nu \mathcal{P}^\mu, \tag{26}$$

$$M^{\mu\nu} = \frac{1}{2}\sqrt{\kappa_0}[p^\mu \mathcal{P}^\nu + p^\mu \mathbb{P}^\nu] - \frac{1}{2\sqrt{\kappa_0}}[x^\mu \mathcal{X}^\nu + x^\mu \mathbb{X}^\nu] + (\mu \leftrightarrow \nu). \tag{27}$$

The above expression for the total angular-momentum in which the orbital angular-momentum contribution $L_{\text{orbital}}^{\mu\nu}$ is isolated, clearly shows that whereas the degrees of freedom $x^\mu = (ct, \vec{x})$ may be interpreted as describing the position of the reciprocal particle in Minkowski spacetime, the dual variables $p^\mu = (E/c, \vec{p})$ play in fact the rôle of internal degrees of freedom which may carry some internal spin structure when properly excited. This separation in the rôles played by the two types of variables $(\mathcal{X}^\mu, \mathcal{P}^\mu)$, namely Q^μ on the one hand, and $(\mathbb{X}^\mu, \mathbb{P}^\mu)$, namely $\mathbb{Q}^\mu$ on the other hand, is to be exploited further later on. Incidentally, one may also write, provided however $\lambda_0 \Pi_\theta \neq 0$,

$$Q^{\mu\nu} = -\mathrm{i}\frac{\sqrt{\kappa_0}}{2\lambda_0 \Pi_\theta}[\overline{Q}^\mu Q^\nu - \overline{\mathbb{Q}}^\mu \mathbb{Q}^\nu], \tag{28}$$

or in real form,

$$L_T^{\mu\nu} = \frac{1}{2\lambda_0 \Pi_\theta}[(\mathcal{X}^\mu \mathcal{P}^\nu - \mathcal{X}^\nu \mathcal{P}^\mu) - (\mathrm{X}^\mu \mathbb{P}^\nu - \mathbb{X}^\nu \mathbb{P}^\mu)], \tag{29}$$

$$M^{\mu\nu} = -\frac{1}{2\lambda_0 \Pi_\theta}\frac{1}{\sqrt{\kappa_0}}[(\mathcal{X}^\mu \mathcal{X}^\nu + \kappa_0 \mathcal{P}^\mu \mathcal{P}^\nu) - (\mathbb{X}^\mu \mathbb{X}^\nu + \kappa_0 \mathbb{P}^\mu \mathbb{P}^\nu)], \tag{30}$$

results which once again display the dual rôles played by the two sectors Q^μ and $\mathbb{Q}^\mu$ which, together with (θ, Π_θ), provide an alternative parametrization of the phase space of the system, namely through the change of variables $\left(x^\mu, \Pi_x^\mu; p^\mu, \Pi_p^\mu; \theta, \Pi_\theta\right) \leftrightarrow (Q^\mu; \mathbb{Q}^\mu; \theta, \Pi_\theta)$.

All these considerations having been made explicit, the evaluation of the algebra of commutation relations for all Noether charges is straightforward. The nonvanishing commutators are found to be

$$[Q^\mu, \overline{Q}^\nu] = 2\hbar \frac{\lambda_0 \Pi_\theta}{\sqrt{\kappa_0}} \eta^{\mu\nu}, \tag{31}$$

$$[Q^{\mu\nu}, Q^\rho] = \mathrm{i}\hbar \eta^{\mu\rho} Q^\nu, \qquad [Q^{\mu\nu}, \overline{Q}^\rho] = -\mathrm{i}\hbar \eta^{\nu\rho} \overline{Q}^\mu, \tag{32}$$

$$[Q^{\mu\nu}, Q^{\rho\sigma}] = \mathrm{i}\hbar[\eta^{\mu\sigma} Q^{\rho\nu} - \eta^{\nu\rho} Q^{\mu\sigma}]. \tag{33}$$

Furthermore the operators Q^μ and $\overline{Q}^\mu$ commute with both $\mathbb{Q}^\mu$ and $\overline{\mathbb{Q}}^\mu$, while one also has

$$[\mathbb{Q}^\mu, \overline{\mathbb{Q}}^\nu] = -2\hbar \frac{\lambda_0 \Pi_\theta}{\sqrt{\kappa_0}} \eta^{\mu\nu}. \tag{34}$$

Written in their real form, these commutation relations also correspond to the algebra of Noether charges,

$$[\mathcal{X}^\mu, \mathcal{P}^\nu] = 2\mathrm{i}\hbar \lambda_0 \Pi_\theta \eta^{\mu\nu}, \qquad [\mathbb{X}^\mu, \mathbb{P}^\nu] = -2\mathrm{i}\hbar \lambda_0 \Pi_\theta \eta^{\mu\nu}, \tag{35}$$

$$\left[L_T^{\mu\nu}, \mathcal{X}^\rho\right] = \mathrm{i}\hbar[\eta^{\mu\rho}\mathcal{X}^\nu - \eta^{\nu\rho}\mathcal{X}^\mu], \qquad \left[L_T^{\mu\nu}, \mathcal{P}^\rho\right] = \mathrm{i}\hbar[\eta^{\mu\rho}\mathcal{P}^\nu - \eta^{\nu\rho}\mathcal{P}^\mu], \tag{36}$$

$$[M^{\mu\nu}, \mathcal{X}^\rho] = \mathrm{i}\hbar\sqrt{\kappa_0}[\eta^{\mu\rho}\mathcal{P}^\nu + \eta^{\nu\rho}\mathcal{P}^\mu], \qquad [M^{\mu\nu}, \mathcal{P}^\rho] = -\mathrm{i}\hbar\frac{1}{\sqrt{\kappa_0}}[\eta^{\mu\rho}\mathcal{X}^\nu + \eta^{\nu\rho}\mathcal{X}^\mu], \tag{37}$$

$$\left[L_T^{\mu\nu}, L_T^{\rho\sigma}\right] = \mathrm{i}\hbar\left[\eta^{\mu\rho}L_T^{\nu\sigma} - \eta^{\mu\sigma}L_T^{\nu\rho} - \eta^{\nu\rho}L_T^{\mu\sigma} + \eta^{\nu\sigma}L_T^{\mu\rho}\right], \tag{38}$$

$$\left[L_T^{\mu\nu}, M^{\rho\sigma}\right] = \mathrm{i}\hbar[\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}], \tag{39}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\hbar\left[\eta^{\mu\rho}L_T^{\nu\sigma} + \eta^{\mu\sigma}L_T^{\nu\rho} + \eta^{\nu\rho}L_T^{\mu\sigma} + \eta^{\nu\sigma}L_T^{\mu\rho}\right]. \tag{40}$$

Note that the $U(D-1,1)$ algebra generated by $Q^{\mu\nu}$ possesses the linear Casimir

$$C_1 = -\mathrm{i}Q^\mu{}_\mu = \frac{1}{2}M^\mu{}_\mu = \sqrt{\kappa_0}p\cdot\Pi_x - \frac{1}{\sqrt{\kappa_0}}x\cdot\Pi_p, \qquad \overline{C}_1 = C_1. \tag{41}$$

One has

$$[C_1, Q^\mu] = \hbar Q^\mu, \qquad [C_1, \overline{Q}^\mu] = -\hbar\overline{Q}^\mu, \qquad [C_1, Q^{\mu\nu}] = 0, \tag{42}$$

or in real form,

$$\begin{aligned} &[C_1, \mathcal{X}^\mu] = \mathrm{i}\hbar\sqrt{\kappa_0}\mathcal{P}^\mu, \qquad [C_1, \mathcal{P}^\mu] = -\mathrm{i}\frac{\hbar}{\sqrt{\kappa_0}}\mathcal{X}^\mu, \\ &\left[C_1, L_T^{\mu\nu}\right] = 0, \qquad [C_1, M^{\mu\nu}] = 0. \end{aligned} \tag{43}$$

Another global symmetry of the dynamics has yet to be addressed, namely its invariance under constant translations in the time evolution parameter τ, $\tau \to \tau + \tau_0$, of which the conserved Noether charge is the canonical Hamiltonian H. A straightforward evaluation of this quantity finds the following identification:

$$H = \frac{1}{2}N_0\left[\overline{\mathbb{Q}}\cdot\mathbb{Q} + \mathbb{Q}\cdot\overline{\mathbb{Q}} + \frac{\lambda_0}{\alpha\kappa_0}\Pi_\theta^2\right]. \tag{44}$$

Since the sectors Q^μ and $\mathbb{Q}^\mu$ commute with one another, this form of the Hamiltonian makes it explicit that indeed the Noether charges Q^μ are conserved. That the Noether charges $Q^{\mu\nu} = z^\mu\Pi^\nu - \overline{z}^\nu\overline{\Pi}^\mu$ are conserved readily follows from a simple direct calculation. In other words, the commutators of all Noether charges with the Hamiltonian do indeed vanish.

As a matter of fact, there exists an alternative representation of the Hamiltonian operator involving the $U(D-1,1)$ linear Casimir C_1. Expressing $\mathbb{Q}^\mu$ in terms of Q^μ, one finds

$$H = \frac{1}{2}N_0\left[\overline{Q}\cdot Q + Q\cdot\overline{Q} + 4\frac{\lambda_0\Pi_\theta}{\sqrt{\kappa_0}}C_1 + \frac{\lambda_0}{\alpha\kappa_0}\Pi_\theta^2\right]. \tag{45}$$

One also has

$$\overline{Q}\cdot Q + Q\cdot\overline{Q} = \mathcal{P}^2 + \frac{1}{\kappa_0}\mathcal{X}^2, \qquad \overline{\mathbb{Q}}\cdot\mathbb{Q} + \mathbb{Q}\cdot\overline{\mathbb{Q}} = \mathbb{P}^2 + \frac{1}{\kappa_0}\mathbb{X}^2. \tag{46}$$

Incidentally, these results imply that the quadratic Casimir operator of the quaplectic algebra $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$ is given by

$$C_2 = \overline{\mathbb{Q}}\cdot\mathbb{Q} + \mathbb{Q}\cdot\overline{\mathbb{Q}}, \tag{47}$$

with the following relation to the linear $U(D-1,1)$ Casimir C_1,

$$C_2 = \overline{Q}\cdot Q + Q\cdot\overline{Q} + 4\frac{\lambda_0\Pi_\theta}{\sqrt{\kappa_0}}C_1 = \mathcal{P}^2 + \frac{1}{\kappa_0}\mathcal{X}^2 + 4\frac{\lambda_0\Pi_\theta}{\sqrt{\kappa_0}}C_1. \tag{48}$$

It is worth noting that the combinations of variables $\left(x^\mu, \Pi_x^\mu, p^\mu, \Pi_p^\mu\right)$ to form the quantities $(\mathcal{X}^\mu, \mathcal{P}^\mu, \mathbb{X}^\mu, \mathbb{P}^\mu)$, and finally $(Q^\mu, \mathbb{Q}^\mu)$, with in particular the Hamiltonian solely expressed in terms of $\mathbb{Q}^\mu$ while the sectors Q^μ and $\mathbb{Q}^\mu$ are commuting with each defining a Heisenberg algebra, is very much reminiscent of the ordinary Landau problem. What corresponds to the Euclidean plane coordinates (x, y) and the magnetic field in the latter case are now, respectively, the pairs of variables (x^μ, p^μ) for each of the spacetime components $\mu = 0, 1, 2, \ldots, D-1$, and the conserved Noether charge $Q_\theta = \Pi_\theta$, except for an overall minus multiplying the contribution of the time component sector $\mu = 0$ to the total Hamiltonian. Indeed, Π_θ is conserved and commutes with all operators contributing to the quantum dynamics. This remark allows one to consider now the diagonalization problem of both the Hamiltonian H and the total energy–momentum $\mathcal{P}^\mu$ of the system.

3.2. *Quantum spectrum: the generic situation* $\Pi_\theta \neq 0$

Since Π_θ is conserved and commutes with all other operators (except θ of course), it is most convenient to consider the diagonalization problem in each of the superselection sectors defined by each of the discrete Π_θ eigenstates $|n\rangle$,

$$\Pi_\theta|n\rangle = \hbar(n+\lambda)|n\rangle, \qquad n \in \mathbb{Z}, \qquad \lambda \in [0, 1[, \tag{49}$$

where λ (defined modulo 1) parametrizes a U(1) holonomy and the freedom in the choice of representation for the Heisenberg algebra $[\theta, \Pi_\theta] = \mathrm{i}\hbar$ associated with the compact degree of freedom $0 \leqslant \theta \leqslant 2\pi$ [17]. In order to exploit the noted analogy with the ordinary Landau problem, let us consider any given superselection sector associated with such a nonvanishing eigenvalue, $\Pi_\theta = \hbar(n+\lambda) \neq 0$. This is guaranteed for all $n \in \mathbb{Z}$ provided $\lambda \neq 0$, or else if $\lambda = 0$ when $n \neq 0$. The particular situation when $\Pi_\theta = 0$ is to be considered separately in section 3.3.

The Π_θ superselection sector having been specified in this manner, for what concerns the remaining variables $\left(x^\mu, \Pi_x^\mu; p^\mu, \Pi_p^\mu\right)$ let us introduce the Fock algebra generators

$$a_+^\mu = \sqrt{\frac{|\lambda_0\Pi_\theta|}{2\hbar\sqrt{\kappa_0}}}\left[\bar{z}^\mu + \mathrm{i}\frac{\sqrt{\kappa_0}}{|\lambda_0\Pi_\theta|}\Pi^\mu\right], \qquad a_+^{\mu\dagger} = \sqrt{\frac{|\lambda_0\Pi_\theta|}{2\hbar\sqrt{\kappa_0}}}\left[z^\mu - \mathrm{i}\frac{\sqrt{\kappa_0}}{|\lambda_0\Pi_\theta|}\overline{\Pi}^\mu\right], \tag{50}$$

$$a_-^\mu = \sqrt{\frac{|\lambda_0\Pi_\theta|}{2\hbar\sqrt{\kappa_0}}}\left[z^\mu + \mathrm{i}\frac{\sqrt{\kappa_0}}{|\lambda_0\Pi_\theta|}\overline{\Pi}^\mu\right], \qquad a_-^{\mu\dagger} = \sqrt{\frac{|\lambda_0\Pi_\theta|}{2\hbar\sqrt{\kappa_0}}}\left[\bar{z}^\mu - \mathrm{i}\frac{\sqrt{\kappa_0}}{|\lambda_0\Pi_\theta|}\Pi^\mu\right]. \tag{51}$$

These operators define two commuting sets of Fock algebras, for each of the spacetime components $\mu, \nu = 0, 1, 2, \ldots, D-1$,

$$\left[a_s^\mu, a_{s'}^{\nu\dagger}\right] = \delta_{s,s'}\eta^{\mu\nu}, \qquad s, s' = +, -. \tag{52}$$

As a function of the sign of the product $\lambda_0\Pi_\theta$, $\eta = \mathrm{sign}\,(\lambda_0\Pi_\theta)$, we then have

$$\text{If } \eta = +1: \quad Q^\mu = \sqrt{\frac{2\hbar|\lambda_0\Pi_\theta|}{\sqrt{\kappa_0}}}\left(-\mathrm{i}a_+^\mu\right), \qquad \mathbb{Q}^\mu = \sqrt{\frac{2\hbar|\lambda_0\Pi_\theta|}{\sqrt{\kappa_0}}}\left(\mathrm{i}a_-^{\mu\dagger}\right); \tag{53}$$

$$\text{If } \eta = -1: \quad Q^\mu = \sqrt{\frac{2\hbar|\lambda_0\Pi_\theta|}{\sqrt{\kappa_0}}}\left(\mathrm{i}a_-^{\mu\dagger}\right), \qquad \mathbb{Q}^\mu = \sqrt{\frac{2\hbar|\lambda_0\Pi_\theta|}{\sqrt{\kappa_0}}}\left(-\mathrm{i}a_+^\mu\right), \tag{54}$$

so that,

$$\Pi^\mu = \frac{1}{2}\left(Q^\mu + \mathbb{Q}^\mu\right) = \mathrm{i}\sqrt{\frac{\hbar|\lambda_0\Pi_\theta|}{2\sqrt{\kappa_0}}}\left[a_-^{\mu\dagger} - a_+^\mu\right], \tag{55}$$

and

$$\mathcal{X}^\mu = \eta\sqrt{\hbar|\lambda_0\Pi_\theta|\sqrt{\kappa_0}}\big[a_\eta^{\mu\dagger} + a_\eta^\mu\big], \qquad \mathcal{P}^\mu = \mathrm{i}\sqrt{\frac{\hbar|\lambda_0\Pi_\theta|}{\sqrt{\kappa_0}}}\big[a_\eta^{\mu\dagger} - a_\eta^\mu\big], \tag{56}$$

while the total angular-momentum reduces to

$$L_T^{\mu\nu} = \frac{1}{2\lambda_0\Pi_\theta}[\mathcal{X}^\mu\mathcal{P}^\nu - \mathcal{X}^\nu\mathcal{P}^\mu] - \mathrm{i}\hbar\big[a_{-\eta}^{\mu}{}^\dagger a_{-\eta}^\nu - a_{-\eta}^{\nu}{}^\dagger a_{-\eta}^\mu\big]. \tag{57}$$

Finally the Hamiltonian is diagonalized in the form

$$H = 2\hbar N_0\frac{|\lambda_0\Pi_\theta|}{\sqrt{\kappa_0}}a_{-\eta}^\dagger\cdot a_{-\eta} + \frac{1}{2}N_0\frac{\lambda_0}{\alpha\kappa_0}\Pi_\theta^2 + \hbar N_0\frac{|\lambda_0\Pi_\theta|}{\sqrt{\kappa_0}}D. \tag{58}$$

Note that the expression (57) shows that the generalized Pauli–Lubanski operator

$$\begin{aligned} W_{\mu_1\mu_2\cdots\mu_{D-3}} &= \frac{1}{2}\epsilon_{\mu_1\mu_2\cdots\mu_{D-3}\mu\nu\rho}\mathcal{P}^\mu L_T^{\nu\rho},\\ S^{\mu\nu\rho} &= \frac{(-1)^{D-3}}{(D-3)!}\epsilon^{\mu\nu\rho\mu_1\mu_2\cdots\mu_{D-3}}W_{\mu_1\mu_2\cdots\mu_{D-3}}, \end{aligned} \tag{59}$$

which characterizes the internal spin representation of quantum states, receives contributions only from the degrees of freedom $\big(a_{-\eta}^\mu, a_{-\eta}^{\mu}{}^\dagger\big)$. The latter are thus to be interpreted as the internal degrees of freedom of the system, while $(\mathcal{X}^\mu, \mathcal{P}^\mu)$ are the spacetime ones commuting with the internal ones, and defining their own Heisenberg algebra, $[\mathcal{X}^\mu, \mathcal{P}^\nu] = 2\mathrm{i}\hbar\lambda_0\Pi_\theta\eta^{\mu\nu} \neq 0$. This interpretation is also consistent with the fact that only the internal degrees of freedom contribute to the spectrum of the Hamiltonian generator of time evolution in τ.

A complete diagonalization, namely the identification of a complete basis of quantum states is thus achieved. Given any of the Π_θ eigenstates with $\Pi_\theta \neq 0$, one takes its tensor product with any of the $\mathcal{P}^\mu$ eigenstates, $|\mathcal{P}^\mu\rangle$, as well as with any of the Fock states associated with the Fock algebra generated by $a_{-\eta}^\mu$ and $a_{-\eta}^{\mu}{}^\dagger$. In the latter sector *a priori* one may have two choices to be contemplated, each with its drawbacks. For the first choice all operators $a_{-\eta}^\mu$ are considered as annihilation operators of the normalized Fock vacuum, with all operators $a_{-\eta}^{\mu}{}^\dagger$ thus being creation operators. In such a situation, the Fock vacuum is Lorentz invariant under the action of $L_T^{\mu\nu}$, and manifest Lorentz covariance of all quantum states is ensured throughout. However, because of the negative definite signature of the time component of the Minkowski metric, $\eta_{00} = -1$, any state involving an odd power of the creation operator $a_{-\eta}^{\mu=0\dagger}$ is of negative norm. The appearance of such negative norm states is generic in any Lorentz covariant quantum system with internal degrees of freedom. Nevertheless, the total number operator contribution to the Hamiltonian, $N = a_{-\eta}^\dagger\cdot a_{-\eta}$, remains then positive definite, with a degeneracy at each level equal to the number of D-partitions of N over the natural numbers, which corresponds to the dimension of the totally N-symmetric representation of $SU(D)$. As an alternative choice avoiding the appearance of these negative norm states, one may wish to use rather $a_{-\eta}^{\mu=0\dagger}$ and $a_{-\eta}^{\mu=i}$ with $i = 1, 2, \ldots, D-1$ as annihilation operators of the Fock vacuum. However such a choice would break Lorentz invariance of the Fock vacuum of the quantized system, a most unwelcome feature. It would also imply an unbounded below spectrum for the quantum Hamiltonian H.

Consequently the choice to be made is the first one which is manifestly Lorentz covariant. It is to be hoped that in a manner similar to what happens in string theory for instance, once the symmetry under translations in the time evolution parameter τ is gauged by coupling the dynamics to a world-line metric in a diffeomorphic invariant manner, the resulting first class constraint is such that these negative norm states are projected out from the physical spectrum, namely that the physical spectrum is restricted to lie within the lowest level $N = 0$ of the

Hamiltonian spectrum. Nonetheless, the fact that the generators of all Poincaré symmetries, namely $\mathcal{P}^\mu$ and $L_T^{\mu\nu}$, commute with the Hamiltonian implies that such a physical spectrum still transforms covariantly under the full Lorentz group, resulting in a consistent physical interpretation of the system independent of the choice of world-line parametrization.

Given the Lorentz covariant choice of Fock states, it follows that the spectrum of the Hamiltonian is organized into discrete levels quite analogous to the Landau levels of the ordinary Landau problem. Each of these excitation levels distinguished by the eigenvalue of the number operator N carries a two-fold degeneracy, one which is finite and corresponds to the D-partitions of N into the natural numbers, and the other which is infinite non-countable and parametrized by the real eigenvalue spectrum of energy–momentum values $\mathcal{P}^\mu$. At any such level $N = 0, 1, 2, \ldots$, the corresponding states are organized into irreducible representations of the full D-dimensional Poincaré group, labeled by the representation of the generalized Pauli–Lubanski tensor (59) generated by the operators $\left(a^\mu_{-\eta}, a^{\mu\,\dagger}_{-\eta}\right)$, as well as the Lorentz invariant measuring the mass-squared value of the states, $(Mc)^2 = -\mathcal{P}^2$. Note that the latter may take its value anywhere in the continuous spectrum defined by the entire real line. Thus for instance even for the lowest states with $N = 0$ which are all of vanishing internal spin and of strictly positive norm, one obtains a continuous mass spectrum with a tachyonic branch for spacelike energy–momenta eigenvalues $\mathcal{P}^\mu$.

3.3. Quantum spectrum: the particular case $\Pi_\theta = 0$

In the Π_θ superselection sector with a vanishing eigenvalue $\Pi_\theta = 0$, most quantities simplify drastically,

$$Q^\mu = \mathbb{Q}^\mu = \Pi^\mu, \qquad \mathcal{P}^\mu = \mathbb{P}^\mu = \Pi^\mu_x, \qquad \mathcal{X}^\mu = \mathbb{X}^\mu = \Pi^\mu_p, \tag{60}$$

so that

$$[\mathcal{X}^\mu, \mathcal{P}^\nu] = 0, \tag{61}$$

as well as

$$H = \frac{1}{2} N_0 \left[\mathcal{P}^2 + \frac{1}{\kappa_0} \mathcal{X}^2 \right], \tag{62}$$

with then

$$\begin{aligned} L_T^{\mu\nu} &= x^\mu \Pi^\nu_x - x^\nu \Pi^\mu_x + p^\mu \Pi^\nu_p - x^\nu \Pi^\mu_p = x^\mu \mathcal{P}^\nu - x^\nu \mathcal{P}^\mu + p^\mu \mathcal{X}^\nu - p^\nu \mathcal{X}^\mu \\ &= L^{\mu\nu}_{\text{orbital}} + p^\mu \mathcal{X}^\nu - p^\nu \mathcal{X}^\mu. \end{aligned} \tag{63}$$

In this case the sector $\left(p^\mu, \Pi^\mu_p\right)$ plays the rôle of internal degrees of freedom, with $\left(x^\mu, \Pi^\mu_x\right)$ that of the spacetime ones. Diagonalization is best achieved in the conjugate momentum basis of $\left(\Pi^\mu_x, \Pi^\mu_p\right) = (\mathcal{P}^\mu, \mathcal{X}^\mu)$ eigenstates. In that case no negative norm states arise in Hilbert space, but the Hamiltonian spectrum is no longer bounded below, since both $\mathcal{P}^2$ and $\mathcal{X}^2$ may take arbitrarily negative eigenvalues, albeit all in a manifestly Lorentz covariant manner. Note that any of the energy–momentum eigenstates of fixed $\mathcal{P}^\mu$ is infinitely non-countably degenerate in the spectrum of $\mathcal{X}^\mu$ for a fixed value of $\mathcal{X}^2$ (when $\Pi_\theta \neq 0$, this infinite degeneracy for a fixed $\mathcal{P}^\mu$ remains discrete). The internal spin representations are those of tensors of all orders characterized by the operator $(p^\mu \mathcal{X}^\nu - p^\nu \mathcal{X}^\mu)$ with $(p^\mu, \mathcal{X}^\mu)$ spanning the Heisenberg algebra $[p^\mu, \mathcal{X}^\nu] = \mathrm{i}\hbar\eta^{\mu\nu}$.

4. World-line quantization and physical states

Coupling the system to a world-line metric in a diffeomorphic invariant way is simple enough. As explained in [16], this is achieved through the action

$$S[x^\mu, p^\mu, \theta; e] = \int \mathrm{d}\tau \left[\frac{1}{e}L + e\Lambda\right], \tag{64}$$

where the dimensionless degree of freedom $e(\tau)$—actually a Lagrange multiplier—is the world-line einbein (using $|e(\tau)|$ rather than $e(\tau)$ in the above action implies invariance also under orientation reversing world-line diffeomorphisms), while L is the Lagrange function in (3). Here, Λ is a real constant of the same physical dimension as L (and H), playing the rôle of a cosmological constant on the world-line [16].

Applying the usual constraint analysis [15] the Hamiltonian formulation of the system is given by

$$S = \int \mathrm{d}\tau\{\dot{x}\cdot\Pi_x + \dot{p}\cdot\Pi_p + \dot{\theta}\Pi_\theta - e(H-\Lambda)\}, \tag{65}$$

H being the Hamiltonian of which the quantization has been considered above. It should be obvious that the present dynamics preserves all the previous global symmetries associated with the quaplectic group, whereas invariance under τ translations has been promoted to a local gauge invariance on the world-line. The first class constraint generating the latter symmetry is the condition

$$\Phi = H - \Lambda = 0, \qquad H = \Lambda, \tag{66}$$

which defines the space of physical states, i.e., states invariant under the world-line diffeomorphism symmetry group. Since all previously identified Noether charges commute with H, indeed quantum physical states fall into representations of the quaplectic group. For physical consistency, the choice of cosmological constant Λ must be such that none of these physical states be of negative norm.

In any Π_θ superselection sector such that $\Pi_\theta = \hbar(n+\lambda) \neq 0$, the $\hat{H}$ eigenspectrum is given by

$$H\colon\quad \frac{1}{2}N_0\frac{\lambda_0}{\alpha\kappa_0}\hbar^2(n+\lambda)^2 + \hbar N_0\frac{|\lambda_0\hbar(n+\lambda)|}{\sqrt{\kappa_0}}\,(2N+D)\,, \tag{67}$$

while we know that no negative norm states are present only in the lowest level $N = 0$. It should be clear that given a choice of values for the parameters κ_0, λ_0 and α, there always exists a range of values for Λ such that a solution to the condition $H = \Lambda$ exists only with $N = 0$ and for some unique choice of λ and n. In this case the gauge invariance constraint projects out from the physical spectrum all negative norm states. We also know that this physical spectrum then includes states of energy–momentum $\mathcal{P}^\mu$ taking an arbitrary value, and such that these are organized into representations of the Poincaré group all of vanishing spin. However, the mass spectrum thereby obtained is not at all constrained, with the invariant mass-squared value $(Mc)^2 = -\mathcal{P}^2$ taking on all real values, hence corresponding to a continuous mass spectrum of massive states, a massless state, and a continuum of spacelike tachyonic states.

In the Π_θ superselection sector with $\Pi_\theta = 0$, namely when $n = 0$ with the choice $\lambda = 0$, no negative norm states arise, and the gauge invariance constraint then reduces to

$$-(Mc)^2 = \mathcal{P}^2 = \frac{2}{N_0}\Lambda - \frac{1}{\kappa_0}\mathcal{X}^2. \tag{68}$$

However, since no restriction applies to the spectra of $\mathcal{X}^\mu$ and $\mathcal{X}^2$ eigenvalues, once again a continuum mass spectrum including a spacelike tachyonic branch is obtained, with a spin content associated with the internal degrees of freedom which spans all possible tensor representations of the corresponding little groups according to whether the states are time-, light- or spacelike.

5. Conclusion

This paper solved the world-line quantization of a simple particle-like system realizing the quaplectic symmetry inherent to the Born–Green reciprocity principle under the exchange of spacetime coordinates $x^\mu = (ct, \vec{x})$ and their conjugate variables $p^\mu = (E/c, \vec{p})$ for a Minkowski spacetime geometry. By considering free motion on the associated Weyl–Heisenberg group, an action invariant under the full quaplectic group $Q(D-1, 1) \cong U(D-1, 1) \ltimes H(D)$, which includes the ordinary Poincaré group as a subgroup, was constructed, and coupled to a world-line geometry in a diffeomorphic invariant manner. Within the quaplectic realization of the Born–Green reciprocity principle, this is indeed the simplest generalization possible of the similar construction for the relativistic scalar particle, in which case the symmetry group over which free motion is considered is the Abelian spacetime translation group. An intriguing outcome of the construction is that the spectrum of the Hamiltonian is organized into Landau-like levels.

The coupling to the world-line geometry involves a constant parameter akin to a world-line cosmological constant. Even though the space of quantum states includes negative norm ones for a manifestly spacetime covariant quantization, the structure of the diffeomorphic gauge invariance constraint is such that for an appropriate choice of the cosmological constant none of the negative norm states belongs to the physical spectrum, in which case the latter are all of zero spin. However, the physical spectrum is infinitely degenerate in the conserved total energy–momentum of the system—the symmetry associated with spacetime translations—with no further restriction on the corresponding Lorentz invariant mass-squared quantity taking all possible real values. Consequently, the physical spectrum consists of relativistic scalar particles but with a continuous mass spectrum including a spacelike tachyonic branch. Given the general scheme of the quaplectic group as a symmetry for reciprocal invariance in phase space, which includes the Poincaré algebra as a subgroup, the appearance of such a continuous mass spectrum certainly seems to be consistent with the conclusions of O'Raifeartaigh's theorem [18].

Some of the features of the conclusions established above may remind one of other work having possibly some relation to Born's reciprocity principle. In [19], based on nonlocal field theories expanded in an appropriate basis of quantum states of internal degrees of freedom of which the dynamics is chosen in an ad hoc manner, a linear discrete mass spectrum is identified. Even though this work bears some resemblance with the ideas of the reciprocity principle, in particular in the manner these were first discussed by Born himself using the relativistic four-dimensional harmonic oscillator, as the author of that work himself states the model of [19] differs from Born's model because of the introduction of these internal degrees of freedom accounting for the nonlocality of fields. Furthermore, the basic equation of [19], which extends the usual massive Klein–Gordon field equation by replacing the mass-squared term by the square of the Hamiltonian of the relativistic harmonic oscillator, is chosen in an ad hoc manner, and does not derive from a more basic principle. As is well known and was recalled previously, the usual massive Klein–Gordon equation may be understood to express the gauge constraint for invariance under local world-line diffeomorphisms. Likewise in our approach the identified wave equation expresses a similar condition for a world-line coupled action invariant under the symmetries of the reciprocal principle, leaving no room for some ad hoc choice of Hamiltonian constraint, except for the value of the cosmological term Λ. At the very best the only point of overlap of our work with that of [19] is with the wavefunctions for the quantum states of the relativistic harmonic oscillator, or more precisely with the configuration space wave representation of the Fock algebra with the signature of the Minkowski metric. But as is well known, in a quantization maintaining Lorentz covariance these states necessarily include negative norm ones, thus leading to negative quantum probabilities, spelling disaster for the theory unless these states may be projected out consistently through some gauge symmetry constraint as done in our analysis. This is also the deficiency of which the work of [20] suffers, even though this may not be obvious since the only aspect in which that latter work may have some relation with Born's reciprocity principle is in the choice of the ground-state wavefunction of the relativistic harmonic oscillator as a basic *ansatz* for the construction of the nucleon electromagnetic form factors. Even though it may appear that the issue of negative norm states is thereby circumvented, it must be pointed out that the corresponding Gaussian quantum state is simply not normalizable: its wavefunction diverges at timelike infinity for the internal degrees of freedom. Except for the empirical interest as a useful parameterization of nucleon form factors in a specific kinematical regime, it would seem difficult to find anything more fundamental in the proposal of [20], however useful it may have been phenomenologically. Since in our work all quantum states remain normalizable in the internal sector, and at worst normalizable in the Dirac sense in the energy–momentum sector, while in the physical sector they are all of positive norm, any attempt at trying to find common points between [20] and the present analysis should prove to be most contrived and artificial indeed.

On the face of it a continuous mass spectrum may seem not to be an appealing feature from a physics point of view. However, it is quite noteworthy that by tuning the value of a simple constant parameter in the action, namely the cosmological constant term Λ, it is possible to project out negative norm states from the physical spectrum, even though the appearance of tachyonic states cannot be avoided. In the case of the ordinary relativistic spinless particle, no negative norm states are possible but this time the cosmological term, which sets the mass of the particle, determines whether the state is time-, light- or spacelike. An appropriate choice for the cosmological term thus excludes the tachyonic solution in that case. In the context of string theory in the critical spacetime dimension, in effect the world-sheet cosmological constant is set to zero because of world-sheet conformal invariance, and indeed all strictly negative norm states are once again projected out from the physical spectrum while spacelike tachyonic states are not necessarily absent on account of world-sheet reparametrization invariance alone. This raises the question whether for two-dimensional quantum gravity there could exist also nonvanishing values of the cosmological constant which would once again project out the negative norm states from the physical spectrum.

One would like to identify possible restrictions on the construction such that both a continuous mass spectrum is avoided or at the least its tachyonic branch excluded. A first naive attempt would be to identify the conserved energy–momentum of the system with the dual conjugate coordinates p^μ, and then in accord with the reciprocity principle, do likewise for the spacetime coordinates x^μ and the conserved quantities associated with translations in the conjugate coordinates. Namely one may wish to impose the following extraneous second-class constraints,

$$\mathcal{P}^\mu = p^\mu, \qquad \mathcal{X}^\mu = x^\mu. \tag{69}$$

However, an interpretation in terms of relativistic states of specific invariant mass and spin requires consistency between the conservation of these quantities and the equations of motion, namely

$$\dot{x}^\mu(\tau) = 0, \qquad \dot{p}^\mu(\tau) = 0. \tag{70}$$

Such restrictions thus prove to be too stringent since they preclude any dynamics whatever.

One may wonder whether, by rendering the construction world-line and/or spacetime supersymmetric, some of the problems could not be avoided. One could indeed hope that a tachyonic branch may be excluded with the help of some spacetime supersymmetric projection of the physical spectrum—as happens with the GSO projection of fermionic strings. However, it is not clear how a continuous mass spectrum, albeit time- and light-like only, could be avoided. One possibility would be to identify a way of compactifying the conserved energy–momentum $\mathcal{P}^\mu$, and hence also $\mathcal{X}^\mu$, on account of Born–Green reciprocity. Since, when $\Pi_\theta \neq 0$, these two quantities do not commute, this would imply not only a discrete mass spectrum, but a finite one as well. Such a full compactification would then render finite the volume of the phase space associated with the conserved quantities $(\mathcal{X}^\mu, \mathcal{P}^\mu)$, namely the generators of the Weyl–Heisenberg subgroup $H(D)$ of the full quaplectic symmetry $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$, of the reciprocally invariant system studied in this work.

However, it is noteworthy to remark that in the context of so-called unparticle physics [21] which has recently spurred so much phenomenological interest, as a matter of fact continuous mass spectra are inherently present [22], leading to unforeseen experimental telltale signs in forthcoming LHC experiments. Besides the tachyonic issue which may possibly be addressed considering supersymmetric extensions of the present analysis, this very point could offer yet a new perspective on the old idea of the Born reciprocity principle and its more modern realizations through dualities of Yang–Mills gauge theories and M-theory [5, 6, 23].

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Appendix

In this appendix, the choice of action (3) used in section 2 is motivated by considering free motion on the Weyl–Heisenberg group associated with the D dimensional Minkowski spacetime coordinates $x^\mu = (ct, \vec{x})$ and their conjugate variables $p^\mu = (E/c, \vec{p})$. Indeed, from the outset such an approach is guaranteed to lead to a system invariant under the full quaplectic group $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$. The Weyl–Heisenberg group is generated by Hermitian operators $\hat{X}^\mu$, $\hat{P}^\mu$ and the unit operator $\mathbb{I}$ such that

$$[\hat{X}^\mu, \hat{P}^\nu] = \mathrm{i}\hbar\eta^{\mu\nu}\mathbb{I}. \tag{A.1}$$

The general Weyl–Heisenberg group element is thus parametrized according to

$$g(\theta, x^\mu, p^\mu) = \mathrm{e}^{\mathrm{i}\theta\mathbb{I}+\frac{\mathrm{i}}{\hbar}p^\mu\hat{X}_\mu-\frac{\mathrm{i}}{\hbar}x^\mu\hat{P}_\mu}, \tag{A.2}$$

where the angular variable θ takes its values, say, in the interval $0 \leqslant \theta \leqslant 2\pi$.

In the case of a general Lie group G of elements g, the G invariant line element on the group manifold is given in the form, at least at a formal level,

$$\mathrm{d}s^2 = -\mathrm{Tr}(g^{-1}\,\mathrm{d}g)^2, \tag{A.3}$$

up to some normalization factor, and the necessity of a proper definition of the trace operation. In turn the action for free motion on such a manifold is of the form, again up to normalization

$$S[g] \propto -\int \mathrm{d}t\,\mathrm{Tr}\left(g^{-1}\frac{\mathrm{d}g}{\mathrm{d}t}\right)^2. \tag{A.4}$$

In the case of the above Weyl–Heisenberg group elements $g(\theta, x^\mu, p^\mu)$ a direct calculation finds

$$\begin{aligned}-\mathrm{Tr}(g^{-1}\,\mathrm{d}g)^2 = {}& \frac{1}{\hbar^2}[\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\mathrm{Tr}\,\hat{P}_\mu\hat{P}_\nu + \mathrm{d}p^\mu\,\mathrm{d}p^\nu\,\mathrm{Tr}\,\hat{X}_\mu\hat{X}_\nu] \\ & + \left[\mathrm{d}\theta - \frac{1}{2\hbar}(\mathrm{d}x^\mu\,p_\mu - x^\mu\,\mathrm{d}p_\mu)\right]^2\,\mathrm{Tr}\,\mathbb{I},\end{aligned} \tag{A.5}$$

in which it is assumed that the definition of the trace operation is such that the operators $\hat{X}^\mu$, $\hat{P}^\mu$, $\hat{X}^\mu\hat{P}^\nu$ and $\hat{P}^\mu\hat{X}^\nu$ are of vanishing trace. Introducing then a regularized definition of the trace operation in the case of the infinite-dimensional representation of the Heisenberg algebra (A.1), it follows that

$$-\mathrm{Tr}(g^{-1}\,\mathrm{d}g)^2 \propto \frac{1}{2}[\mathrm{d}x^\mu\,\mathrm{d}x_\mu + \mu_0\,\mathrm{d}p^\mu\,\mathrm{d}p_\mu] + \frac{1}{2}\mu_1\left[\mathrm{d}\theta - \frac{1}{2\hbar}(\mathrm{d}x^\mu\,p_\mu - x^\mu\,\mathrm{d}p_\mu)\right]^2, \tag{A.6}$$

where $\mu_0 > 0$ and $\mu_1 \geqslant 0$ are regularization-dependent normalization factors of the appropriate physical dimension. This latter expression provides the choice of action considered in section 2.

An alternative realization of the Weyl–Heisenberg group is provided by the following $(2D+2)\times(2D+2)$ real matrices [12],

$$\begin{aligned} H(\theta, x^\mu, p_\mu) &= \begin{pmatrix} \mathbb{I}_D & 0 & 0 & p_\mu \\ 0 & \mathbb{I}_D & 0 & x^\mu \\ -x^\mu & p_\mu & 1 & 2\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{with} \\ \mathrm{d}H(\theta, x^\mu, p_\mu) &= \begin{pmatrix} 0 & 0 & 0 & \mathrm{d}p_\mu \\ 0 & 0 & 0 & \mathrm{d}x^\mu \\ -\mathrm{d}x^\mu & \mathrm{d}p_\mu & 0 & 2\mathrm{d}\theta \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{A.7}$$

where $x^\mu, p_\mu \in \mathbb{R}$ and $0 \leqslant \theta < 2\pi$. The line element $\mathrm{d}\ell^2 = \frac{1}{4}\mathrm{Tr}(H^{-1}\mathrm{d}H)^2$ is thus

$$\mathrm{d}\ell^2 = \tfrac{1}{2}\mathrm{d}x^\mu\,\mathrm{d}x_\mu + \tfrac{1}{2}\,\mathrm{d}p^\mu\,\mathrm{d}p_\mu + \tfrac{1}{2}[\mathrm{d}\theta - (\mathrm{d}x^\mu\,p_\mu - x^\mu\,\mathrm{d}p_\mu)]^2, \tag{A.8}$$

in which the Minkowski metric is used to raise or lower indices where necessary. Up to scalings of θ and p_μ relative to x^μ by appropriate dimensional constants, this expression reproduces the required line element for our world-line action.

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